In the first lesson, we covered the definition of a complex number, some basic terminology, the complex plane, and addition and subtraction of complex numbers. In this lesson, we'll cover the operations of multiplication and division.

Multiplication of complex numbers is not as simple as addition, but still fairly straight forward. We can calculate the product of two complex numbers by multiplying as binomials using distributive property.

$$\begin{aligned} &(a + bi)(c + di) \\ =&a(c + di) + bi(c + di) \\ =&ac + adi + bci + bdi^2 \\ =&(ac - bd) + (bc + bd)i \end{aligned}$$

Showing that two complex numbers mutiplied is equal to the real parts multiplied minus the imaginary parts multiplied, plus the real parts of each multiplied by the imaginary parts of the other times i ($zw = (Re(z)Re(w) - Im(z)Im(w)) + (Re(z)Im(w) + Re(w)Im(z))i$).

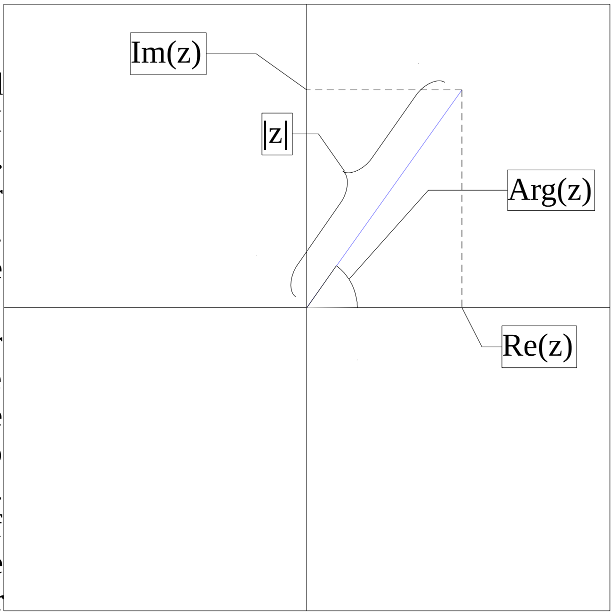
One important property of complex multipication is that a complex number times its conjugate is always a real number. This can easily be shown:

$$\begin{aligned} &(a + bi)(a - bi) \\ =&a(a - bi) + bi(a - bi) \\ =&a^2 - abi + abi - b^2i^2 \\ =&a^2 + b^2 \end{aligned}$$

Complex division, on the other hand, is not as intuitive. However, one can use the above property to divide complex numbers by multiplying both the numerator and the denominator by the conjugate of the denominator:

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)\cdot(c-di)}{c^2+d^2}$$

$$\frac{z}{w} = \frac{z\bar{w}}{\Re(w)^2 + \Im(w)^2}$$



This can be reduced by dividing both the real and imaginary parts by the denominator. And yes, those fancy R and I things in the denominator mean the same thing as *Re()* and *Im()*. I'm not sure if I'll be able to consistently use one or the other because I will sometimes be using LaTeX and sometimes not. Sorry for that inconvenience, just remember that they're the same thing.

Now, because that lesson was so much freaking shorter than I expected, I'm combining lessons 2 and 3. In lesson 1, we introduced the complex plane, which allows for a simple graphical representation of addition of complex numbers. It also shows many important geometric properties of complex numbers. We can represent a complex number by the x and y coordinates of the complex number in the complex plane, which represent the real and imaginary parts of the number, *Re(z)* and *Im(z)*. Another way of representing a complex number is by its distance from the origin and the angle it makes with the x-axis, known respecitvely as the magnitude and argument of *z*, |*z*| and *Arg(z)*, as demonstrated in this diagram.

The magnitude of a complex number is simple enough to calculate, form pythagoreon's thereom we can see that it is the hypotenuse of a right triangle with the sides of the magnitude, the real part and the imaginary part, and therefore $|z|=\sqrt{\Re(z)^2+\Im(z)^2}=\sqrt{z\bar{z}}$ from the property we earlier explored that $z\bar{z}=\Re(z)^2+\Im(z)^2$.

The argument can be calculated using some trigonometry. Since the tangent function gives us the angle from the side length of the side of the triangle opposite the angle divided by that of the adjacent side, and the opposite and adjacent sides of this triangle relative to the argument of *z* are equal to the real and imaginary parts of z, the formula for the argument of *z* is:

$$Arg(z)=\tan^{-1}\left(\frac{\Im(z)}{\Re(z)}\right)$$

We can describe a complex number in polar form, which is its representation by the magnitude and argument rather than by its real and imaginary parts.

$|z|(\cos(Arg(z))+\sin(Arg(z))i)$

This holds true because cosine of an angle gives the ratio of the adjacent side over the hypotenuse, in this case the real part divided by the magnitude. The magnitude is then multiplied from outside the brackets to cancel thie denominator ratio out and reduce the fraction to the numerator, in this case the real part. Similarly, sine of the argument gives the imaginary part over the magnitude, which multiplied by the magnitude gives the imaginary part. In the same way that $Re(z) + Im(z)i$ is often written as $a + bi$ so as to be more readable, $|z|(\cos(Arg(z))+\sin(Arg(z))i)$ is usually written as $r(\cos(\theta)+\sin(\theta)i)$.

There are four formulas known as the trigonometric sum identities that hold for any and all x and y:

$\cos(x)\cos(y)-\sin(x)\sin(y) = \cos(x+y)$
$\cos(x)\cos(y)+\sin(x)\sin(y) = \cos(x-y)$
$\cos(x)\sin(y)+\cos(y)\sin(x) = \sin(x+y)$
$\cos(x)\sin(y)-\cos(y)\sin(x) = \sin(x-y)$

As is turns out, when we multiply two complex numbers in polar form $r(\cos(x)+\sin(x)i)$ and $s(\cos(y)+\sin(y)i)$, the product is $rs(\cos(x)\cos(y)-\sin(x)\sin(y)+(\cos(x)\sin(y)+\cos(y)\sin(x))i)$, whic from the above formulas reduces to $rs(\cos(x+y)+\sin(x+y)i)$. This means that to multiply two complex numbers in polar form, you add the angles and multiply the magnitudes. Because of my low level diagraming skills I'm not really going to attempt to draw this out because I'd have to draw a scale and everything, but this property will be important for our next lesson, for which I'll find a way to illustrate things more sufficiently.

Optional exercises time!

1. Multiply.
    - a) $(5+4i)(3-4i)$
    - b) $(-1-2i)(-3+2i)$
2. Divide.
    - a) $(1-3i)/(2+2i)$
    - b) $(-7+8i)/(9+10i)$
3. Put into polar form.
    - a) $5+5i$
    - b) $5-7i$

4. Multiply. Just using degrees instead of radians for now.
    - a) $2(\cos(45)+\sin(45)i)3(\cos(45)+\sin(45)i)$
    - b) $5(\cos(15)+\sin(15)i)10(\cos(60)+\sin(60)i)$
    - c) $1(\cos(30)+\sin(30)i)1(\cos(25)+\sin(25)i)$
    - d) $4(\cos(20)+\sin(20)i)4(\cos(50)+\sin(50)i)$

Yea multiplying in polar form is pretty easy, but it's important to understand.